Bayesian excursion set estimation with GPs

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Outline

Outline

The framework

In this talk we focus on the problem of determining the set

$$\Gamma^\star = \{x \in D : f(x) \in T\} = f^{-1}(T)$$

where $D \subset \mathbb{R}^d$ is compact, $f : D \longrightarrow \mathbb{R}^k$ is measurable, $T \subset \mathbb{R}^k$.

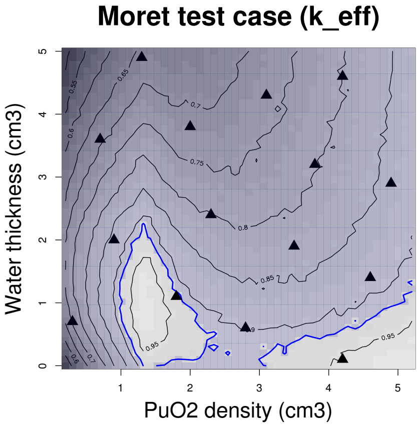
Here: $k = 1$, f is continuous, and $T = (-\infty, t]$ for a fixed $t \in \mathbb{R}$.

$\Gamma^\star = \{x \in D : f(x) \leq t\}$ is denoted the excursion set of f below t.

Objective

Estimate $\Gamma^\star$ and quantify uncertainty on it when f is evaluated only at a few points $\mathbf{X}_n = \{x_1, \ldots, x_n\} \subset D$.

The framework: IRSN test case



Test case:

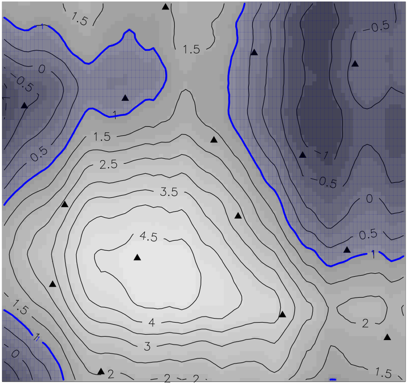
- $\mathrm{k_{eff}}$ function of $\mathrm{PuO_2}$ density and $\mathrm{H_2O}$ thickness, $D = [0.2, 5.2] \times [0, 5]$;
- continuous function, expensive to evaluate;
- $n = 20$ observations (black triangles);

Objective: estimate $\Gamma^\star = \{x \in D : f(x) \leq t\}$ and evaluate the uncertainty of the estimate.

Acknowledgements: Yann Richet, Institut de Radioprotection et de Sûreté Nucleaire.

The framework: an example

Gaussian random field realization



Function $f : D \subset \mathbb{R}^d \to \mathbb{R}$

- expensive to evaluate;
- continuous.

Evaluated at
$\mathbf{X}_n = (x_1, \dots, x_n)$ (black triangles)
with values $\mathbf{f}_n = (f(x_1), \dots, f(x_n))$.

Objective: estimate
$\Gamma^\star = \{x \in D : f(x) \leq t\}$ and
evaluate the uncertainty of the estimate.

Bayesian approach

Bayesian framework: f *is seen as one realization of a (GRF)* $(Z_x)_{x \in D}$ with prior mean m and covariance kernel k.

Given the function evaluations $\mathbf{f}_n$ the posterior field has a Gaussian distribution

$$Z \mid (Z(\mathbf{X}_n) = \mathbf{f}_n)$$

with mean and covariance kernel

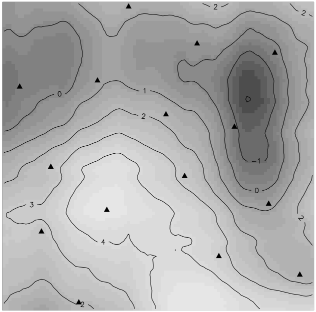
$$m_n(x) = m(x) + k(x, \mathbf{X}_n) k(\mathbf{X}_n, \mathbf{X}_n)^{-1} (\mathbf{f}_n - m(\mathbf{X}_n))$$
$$k_n(x, y) = k(x, y) - k(x, \mathbf{X}_n) k(\mathbf{X}_n, \mathbf{X}_n)^{-1} k(\mathbf{X}_n, y)$$

$\Gamma^\star$ is a realization of $\Gamma = \{x \in D : Z_x \leq t\} = Z^{-1}((-\infty, t])$

A prior on the space of functions

Assume: f realization of $(Z_x)_{x \in D}$, Gaussian Random Field (GRF)

Posterior GRF realization



Prior: $(Z_x)_{x \in D}$ with

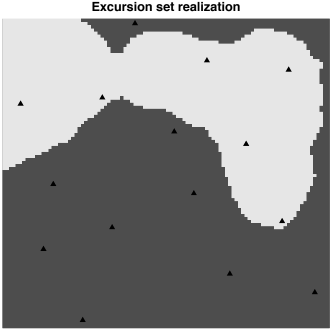
- a.s. continuous paths;
- Matérn covariance kernel k $(\nu = 3/2)$;
- constant mean function m.

Given $n = 15$ evaluations $\mathbf{f}_n$ at $\mathbf{X}_n$

Posterior field: $Z \mid Z_{\mathbf{X}_n} = \mathbf{f}_n$
with mean m_n and covariance k_n.

Distribution of excursion sets

The posterior field defines posterior distribution on excursion sets.
$\Gamma = \{x \in D : Z_x \leq t\}$



How to summarize the distribution on sets?

The posterior excursion set is a random closed set.

Here we focus on **Expectations** of random closed sets[1]

- Vorob'ev expectation
- distance average expectation

Conservative estimates, based on Vorob'ev quantiles.

1. for more definitions of expectation see Molchanov, I. (2005). Theory of Random Sets. Springer.

Main references:

E. Vazquez and M. P. Martinez. (2006). *Estimation of the volume of an excursion set of a Gaussian process using intrinsic kriging.* Tech Report. arXiv:math/0611273.

Ranjan, P., Bingham, D., and Michailidis, G. (2008). *Sequential experiment design for contour estimation from complex computer codes.* Technometrics, 50(4):527541.

Bect, J., Ginsbourger, D., Li, L., Picheny, V., and Vazquez, E. (2012). *Sequential design of computer experiments for the estimation of a probability of failure.* Stat. Comput., 22 (3):773793.

Chevalier, C., Bect, J., Ginsbourger, D., Vazquez, E., Picheny, V., and Richet, Y. (2014). *Fast kriging-based stepwise uncertainty reduction with application to the identification of an excursion set.* Technometrics.

Chevalier, C., Ginsbourger, D., Bect, J., and Molchanov, I. (2013). *Estimating and quantifying uncertainties on level sets using the Vorobev expectation and deviation with Gaussian process models.* mODa 10.

Bolin, D. and Lindgren, F. (2015), French, J. P. and Sain, S. R. (2013)

and references therein...

Outline

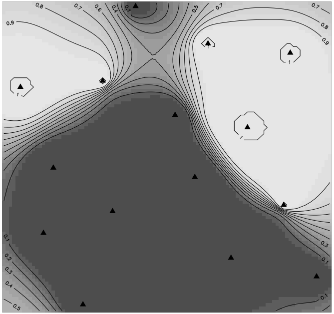
Vorob'ev quantiles

The function

$$p_n : x \in D \to p_n(x) = P_n(x \in \Gamma) \in [0, 1]$$

is the **coverage function of** Γ, where $P_n(\cdot) = P(\cdot \mid Z_{\mathbf{X}_n} = \mathbf{f}_n)$.

Coverage probability function

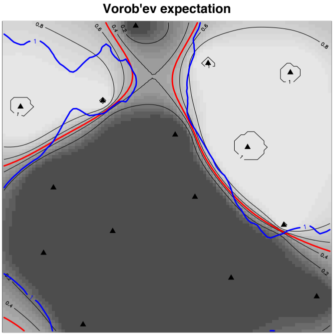


In the Gaussian case

- fast to compute
  $p_n(x) = \Phi\left(\frac{m_n(x) - t}{\sqrt{k_n(x,x)}}\right)$
- marginal statement
- creates a family of set estimates
  $Q_\rho = \{x \in D : p_n(x) \geq \rho\}$

Vorob'ev expectation

Consider a Borel measure μ on D. From the family of "quantiles" Q_ρ we can choose $Q_{\widetilde{\rho}}$ such that $\mu(Q_{\widetilde{\rho}}) = \mathbb{E}[\mu(\Gamma)]$.



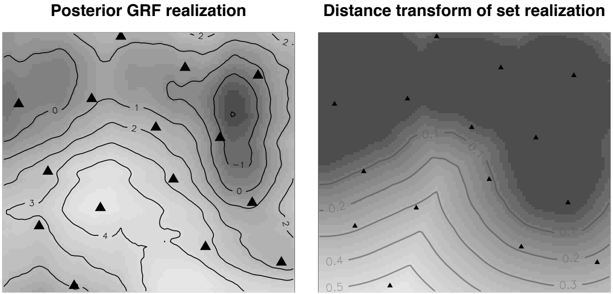
Properties:

- based on the measure μ;
- for some choices of μ, fast to compute;
- no confidence statements on the set.

Chevalier, C., Ginsbourger, D., Bect, J., and Molchanov, I. (2013). *Estimating and quantifying uncertainties on level sets using the Vorobev expectation and deviation with Gaussian process models.* mODa 10

Distance average approach

Consider the distance function $d : (x, \Gamma) \to d(x, \Gamma)$.



Γ is random therefore $d(x, \Gamma)$ is a random variable for each $x \in D$.

Distance average expectation

Given the distance function $d(x, \Gamma)$, the expected distance function

$$\overline{d}(x) = \mathbb{E}[d(x, \Gamma)]$$

The distance average expectation of Γ is the set

$$\mathbb{E}_{DF}[\Gamma] = \{x \in D : \overline{d}(x) \leq \overline{\varepsilon}\} \quad \text{where}$$

$\overline{\varepsilon}$ is chosen in order to obtain a distance function for the set $\mathbb{E}_{DF}[\Gamma]$ as "close" as possible to $\overline{d}$ in a L^2 sense.

An **uncertainty assessment** for the estimate is

$$\mathrm{DFV}_{\Gamma} = \mathbb{E}\|\overline{d}(\cdot) - d(\cdot, \Gamma)\|_2^2$$

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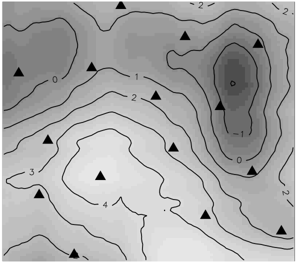
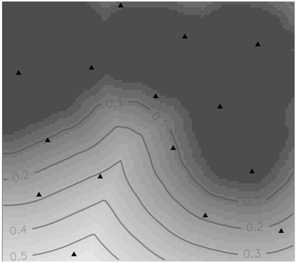
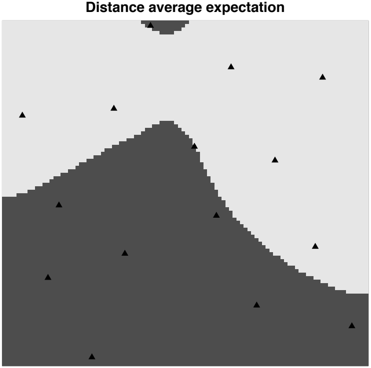


Distance transform of set realization



For each realization of Γ (expensive!) we compute the distance function and then consider an average over the functions.

Distance average expectation



$$\mathbb{E}_{DF}[\mathit{\Gamma}] = \{x \in D : \overline{d}(x) \leq \overline{\varepsilon}\}$$

Outline

An approximate Gaussian random field

Assumption: the GRF Z has been evaluated at $x_1, \ldots, x_n \in D$.

We denote by $Z_E = (Z_{e_1}, \ldots, Z_{e_m})'$ the random vector of values of Z at $E = \{e_1, \ldots, e_m\} \subset D$.

Here we focus on affine predictors of Z of the form

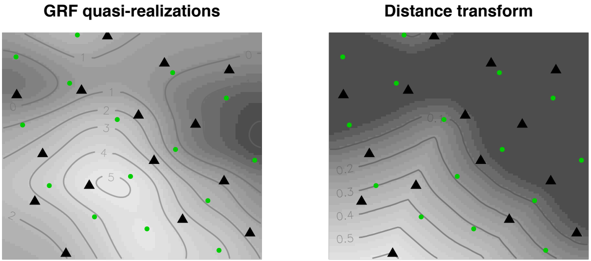
$$\widetilde{Z}_x = a(x) + \mathbf{b}^T(x) Z_E \quad (x \in D),$$

where $a : D \longrightarrow \mathbb{R}$ is a trend function and $\mathbf{b} : D \longrightarrow \mathbb{R}^m$ is a vector-valued function of deterministic weights.

Similarly, we approximate Γ by the excursion set of $\widetilde{Z}$:
$\widetilde{\Gamma} = \{x \in D : \widetilde{Z}_x \leq t\}$

Towards an optimal design of simulation points

The simulation points E could be chosen with a LHS design ($m = 30$)



However, we do not control on how close is Γ to $\widetilde{\Gamma}$

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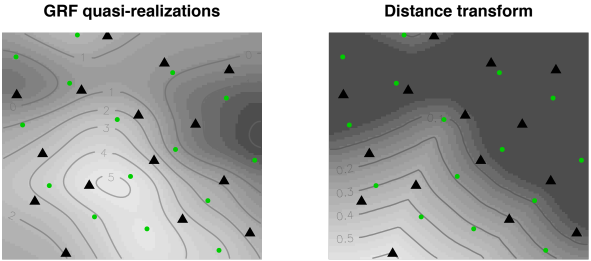

However, we do not control on how close is Γ to $\widetilde{\Gamma}$

What distance between Γ to $\widetilde{\Gamma}$?

Definition: the function

$$(\Gamma_1, \Gamma_2) \in D \times D \longrightarrow d_{\mu,n}(\Gamma_1, \Gamma_2) = \mathbb{E}[\mu(\Gamma_1 \Delta \Gamma_2) \mid Z_{\mathbf{x}_n} = \mathbf{f}_n]$$

is called expected distance in measure between Γ_1, Γ_2.

Proposition: distance in measure between Γ and $\widetilde{\Gamma}$

a) If Z and $\widetilde{Z}$ are random fields such that Γ and $\widetilde{\Gamma}$ are random closed sets, $D \subset \mathbb{R}^d$ is compact and μ is a finite Borel measure on D, we have

$$d_{\mu,n}(\Gamma, \widetilde{\Gamma}) = \int \rho_{n,m}(x)\mu(dx) \text{ where}$$

$$\rho_{n,m}(x) = P_n(x \in \Gamma \Delta \widetilde{\Gamma}) = \mathbb{P}_n(Z_x \geq t, \widetilde{Z}_x < t) + \mathbb{P}_n(Z_x < t, \widetilde{Z}_x \geq t)$$

Towards an optimal design of simulation points

Proposition: distance in measure between Γ and $\widetilde{\Gamma}$

b) If Z is Gaussian conditionally on $Z_{\mathbf{X}_n}$ with conditional mean $\mathfrak{m}_n$ and conditional covariance kernel k_n, we get

$$\mathbb{P}_n(Z_x \geq t, \widetilde{Z}_x < t) = \Phi_2\left(\mathbf{c}_n(x, E), \Sigma_n(x, E)\right),$$

with $\mathbf{c}_n(x, E) = \begin{pmatrix} \mathfrak{m}_n(x) - t \\ t - a(x) - \mathbf{b}(x)^T \mathfrak{m}_n(E) \end{pmatrix}$

and $\Sigma_n(x, E) = \begin{pmatrix} k_n(x, x) & -\mathbf{b}(x)^T k_n(E, x) \\ -\mathbf{b}(x)^T k_n(E, x) & \mathbf{b}(x)^T k_n(E, E)\mathbf{b}(x) \end{pmatrix}$

Towards an optimal design of simulation points

c) Particular case: If $\widetilde{Z}$ is chosen as best linear unbiased predictor of Z given $Z(\mathbf{X}_n)$, then $\mathbf{b}(x) = k_n(E,E)^{-1}k_n(E,x)$ so that

$$\Sigma_n(x,E) = \begin{pmatrix} k_n(x,x) & -\gamma_n(x,E) \\ -\gamma_n(x,E) & \gamma_n(x,E) \end{pmatrix}$$

where $\gamma_n(x,E) = \mathrm{Var}_n[\widehat{Z}_x] = k_n(E,x)^T k_n(E,E)^{-1} k_n(E,x)$.

Optimal design(s) of simulation points can be obtained by minimizing

$$d_{\mu,n}(\Gamma,\widetilde{\Gamma}(E)) = \int \Phi_2\left(\mathbf{c}_n(x,E),\Sigma_n(x,E)\right) + \Phi_2\left(-\mathbf{c}_n(x,E),\Sigma_n(x,E)\right)\mu(\mathrm{d}x)$$

over $(\mathbf{e}_1,\ldots,\mathbf{e}_m) \in D^m$.

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over $(\mathbf{e}_1, \ldots, \mathbf{e}_m) \in D^m$.

Procedure overview

Approximate Z at each point with

$$\widetilde{Z}_x = a(x) + \mathbf{b}^T(x) Z_E \text{ with } E = \{\mathbf{e}_1, \dots, \mathbf{e}_m\}$$

The points in E are chosen with one of the following algorithms:

Algorithm A (Full criterion): sequential minimization of

$$d_{\mu,n}(\Gamma, \widetilde{\Gamma}(E_i^*)) = \int \Phi_2 \left(\mathbf{c}_n(x, E_i^*), \Sigma_n(x, E_i^*)\right) + \Phi_2 \left(-\mathbf{c}_n(x, E_i^*), \Sigma_n(x, E_i^*)\right) \mu(\mathrm{d}x)$$

with respect to $\mathbf{e}_i$ where $E_i^* = \{\mathbf{e}_1^*, \dots, \mathbf{e}_{i-1}^*\} \cup \{\mathbf{e}_i\}$;

Algorithm B (Fast heuristic): sequential maximization of

$$\rho_{n,E}(x) = \Phi_2 \left(\mathbf{c}_n(x, E), \Sigma_n(x, E)\right) + \Phi_2 \left(-\mathbf{c}_n(x, E), \Sigma_n(x, E)\right)$$

with respect to x;

Procedure overview

Approximate Z at each point with

$$\widetilde{Z}_x = a(x) + \mathbf{b}^T(x) Z_E \text{ with } E = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$$

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with respect to $\mathbf{e}_i$ where $E_i^* = \{\mathbf{e}_1^*, \ldots, \mathbf{e}_{i-1}^*\} \cup \{\mathbf{e}_i\}$;

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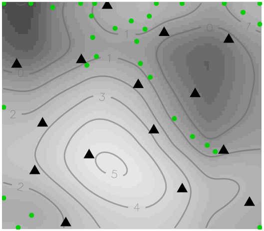
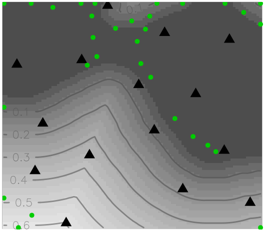
with respect to x;

Quasi realizations for distance average variability

GRF quasi-realizations



Distance transform



A.D. and Bect, J. and Chevalier, C. and Ginsbourger, D. (2016) *Quantifying uncertainties on excursion sets under a Gaussian random field prior*. SIAM/ASA J.Uncertainty Quantification, 4(1):850–874. hal-01103644v2.

Test case: negative Branin-Hoo function

Quantity of interest: $\mathrm{DTV}_{\Gamma,N} = \frac{1}{N}\sum_{i=1}^{N}\|d_N^*(\cdot) - d(\cdot,\Gamma_i)\|_2^2$

Experimental set-up:

- 20 observation points;
- $N = 10000$ conditional simulations on a 50×50 grid;
- $K = 100$ replications of each experiment.

Methods:

1. Full Monte Carlo simulations on the grid,
2. Simulations at optimized points (A,B) and interpolation on the same grid.

Test case: negative Branin-Hoo function

Method 1: Full grid simulations

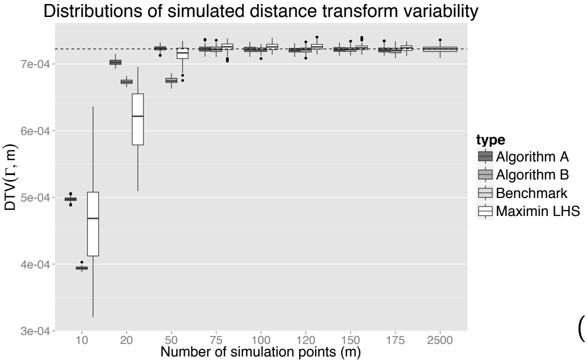
- Variability:

$$\widehat{\mathbb{E}}\left[\mathrm{DTV}_{\Gamma,10000}\right] = 7.2 \times 10^{-4} (\pm 4.71 \times 10^{-8});$$
$$\widehat{\mathrm{Var}}\left[\mathrm{DTV}_{\Gamma,10000}\right] = 2.22 \times 10^{-11} (\pm 3.14 \times 10^{-13});$$

- total computational cost: **10498 seconds**.

Test case: negative Branin-Hoo function

Method 2: quasi-realizations on 50×50 grid.



Total computing cost

- Algorithm A, $m = 150$: **11201 sec** (10566 for simulation point optimization)
- Algorithm B, $m = 150$: **812 sec** (250 for simulation point optimization)
- LHS, $m = 150$: **691 sec**

(Benchmark: **10498 sec**)

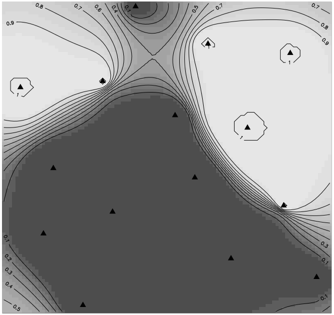
Outline

Vorob'ev quantiles

The coverage function $p_n : x \to p_n(x) = P_n(x \in \Gamma)$ defines the family of set estimates

$$Q_\rho = \{x \in D : p_n(x) \geq \rho\}$$

Coverage probability function

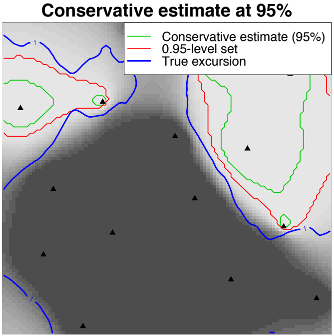


If $\rho = \widetilde{\rho}$ we have Vorob'ev expectation

High values of ρ gives us sets with high marginal probability of observing the set.

A **conservative estimate** of Γ^* is

$$C_{\Gamma,n} = Q_{\rho^*} \text{ where } \rho^* \in \arg\max_{\rho\in[0,1]}\{\mu(Q_\rho) : P_n(Q_\rho \subset \{Z_x \leq t\}) \geq \alpha\}$$



- joint confidence statement on the set estimate;
- method introduced for Gauss Markov random fields;
- expensive to compute otherwise.

Bolin, D. and Lindgren, F. (2015). *Excursion and contour uncertainty regions for latent Gaussian models.* JRSS: B, 77(1):85-106.

The computation of conservative estimates

The family of sets Q_ρ is *nested*, therefore we can obtain $C_{\Gamma,n}$ with a **dichotomy on the level** ρ.

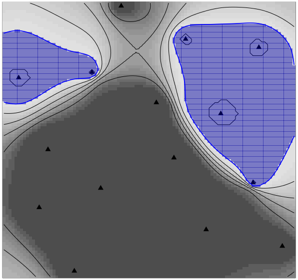
At each iteration of the dichotomy we need to compute

$$P_n(Q_\rho \subset \{Z_x \leq t\}) = P_n(Z_{e_1} \leq t, \dots, Z_{e_k} \leq t),$$

where $E = \{e_1, \dots, e_k\}$ is a the discretization of Q_ρ.

- randomized quasi Monte Carlo integration by Genz et al. :
  (Fast, reliable, dimension dependent, available only $k < 1000$)
- standard Monte Carlo.
  (dimension independent, many samples for low variance)

A quasi Monte Carlo algorithm for orthant probabilities



$$P_n(Q_\rho \subset \{Z_x \leq t\}) =$$
$$P_n(Z_{e_1} \leq t, \ldots, Z_{e_k} \leq t) =$$
$$1 - P_n(\max_{x \in E} Z_x > t) = 1 - p$$

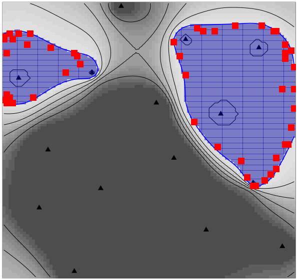
Main idea: $p = P_n(\max_E Z_x > t) = p_q + (1 - p_q) R_q$, where

$$p_q = P_n(\max_{E_q} Z_x > t), \qquad R_q = P_n(\max_{E \setminus E_q} Z_x > t \mid \max_{E_q} Z_x \leq t).$$

Genz algorithm (QRSVN) — Monte Carlo methods

A quasi Monte Carlo algorithm for orthant probabilities

Candidate $\rho = 0.95$



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$$p_q = P_n(\max_{E_q} Z_x > t),$$
Genz algorithm (QRSVN)

$$R_q = P_n(\max_{E \setminus E_q} Z_x > t \mid \max_{E_q} Z_x \leq t).$$
Monte Carlo methods

$$\widehat{p_q} = 0.47 \qquad \widehat{R_q} = 0.42 \qquad \Rightarrow \widehat{p} = 0.69$$

Computation of the remainder

$R_q = P_n(\max_{E \setminus E_q} Z_x > t \mid \max_{E_q} Z_x \leq t)$

Standard Monte Carlo:

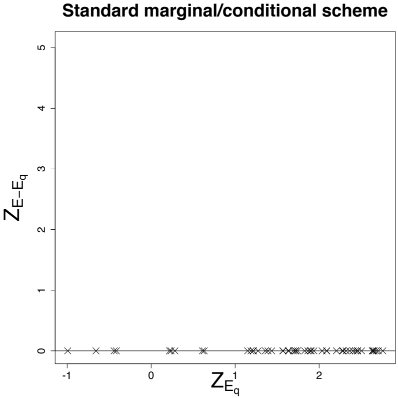
1. draw realizations $z_1^q, \ldots, z_s^q$ from $Z_{E_q} \mid \max_{E_q} Z_x \leq t$;
2. for each z_i^q, draw a realization from $Z_{E \setminus E_q} \mid Z_{E_q} = z_i^q$;
3. Estimate R_q with $R_q^{\text{MC}} = \frac{1}{s} \sum_{i=1}^{s} \mathbf{1}_{\max(Z_{E \setminus E_q}(\omega_i) \mid Z_{E_q} = z_i^q) > t}$

The cost of step 1 is higher than the cost of step 2.

At fixed computational budget we reduce the variance of R_q^{MC} exploiting this difference with **asymmetric nested Monte Carlo**.

Computation of the remainder

At fixed computational budget we reduce the variance of R_q^{MC} drawing many realizations of $Z_{E \setminus E_q} \mid Z_{E_q} = z_i^q$ for each z_i.



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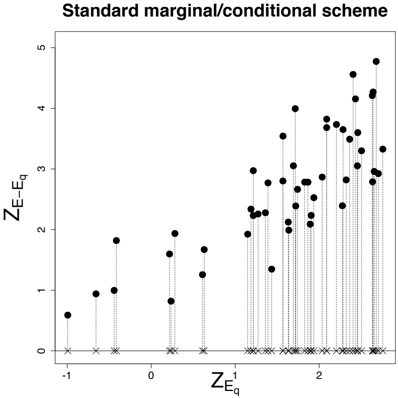

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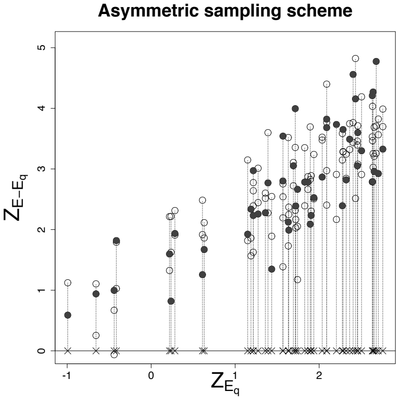


Computation of the remainder: asymmetric nested MC

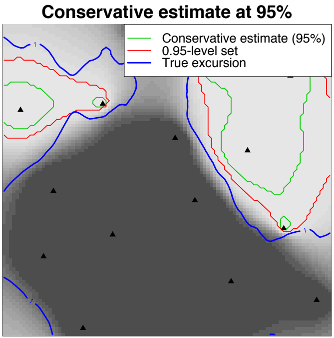
$R_q = P_n(\max_{E \setminus E_q} Z_x > t \mid \max_{E_q} Z_x \leq t)$

1. *draw realizations* $z_1^q, \ldots, z_s^q$ *from* $Z_{E_q} \mid \max_{E_q} Z_x \leq t$;
2. *for each* z_i^q, **draw** $m^* > 1$ **samples** from $Z_{E \setminus E_q} \mid Z_{E_q} = z_i^q$;
3. $R_q^{\mathrm{anMC}} = \frac{1}{s} \frac{1}{m^*} \sum_{i=1}^{s} \sum_{j=1}^{m^*} \mathbf{1}_{\max(Z_{E \setminus E_q}(\omega_{i,j}) \mid Z_{E_q} = z_i^q) > t}$

$\mathrm{var}(R_q^{\mathbf{anMC}})$ **is optimally reduced if:** $m^* = \sqrt{\frac{(\alpha+c)B}{\beta(A-B)}}$,
where $A = \mathrm{var}(\mathbf{1}_{\max(Z_{E \setminus E_q} \mid Z_{E_q}) > t})$, $B = \mathbb{E}\left[\mathrm{var}(\mathbf{1}_{\max(Z_{E \setminus E_q} \mid Z_{E_q}) > t} \mid \max_{E_q} Z_x \leq t)\right]$ and α, β, c system dependent constants.

A D. and Ginsbourger D. (2016). *Estimating orthant probabilities of high dimensional Gaussian vectors with an application to set estimation*. Submitted, hal-01289126

Comparison with standard Monte Carlo

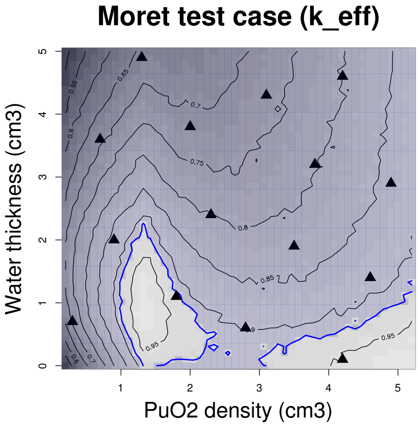


Full discretization: grid 100×100

Time for equivalent estimates:

- Full MC: 1520 seconds;
- GMC: 200 seconds;
- GanMC: 136 seconds.

The IRSN test case



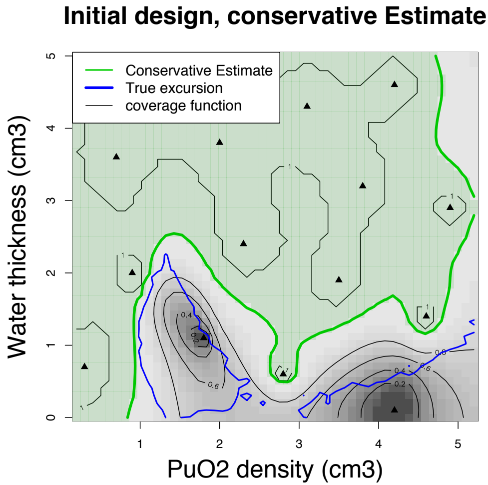
Test case:

- $\mathrm{k}_{\mathrm{eff}}$ function of PuO_2 density and $\mathrm{H}_2\mathrm{O}$ thickness, $D = [0.2, 5.2] \times [0, 5]$;
- $n = 20$ observations (LHS design);
- $\Gamma^* = \{x \in D : \mathrm{k}_{\mathrm{eff}}(x) \leq t\}$, $t = 0.92$

GRF model

- constant prior mean, Matérn ($\nu = 5/2$) covariance;
- MLE for parameters.

Acknowledgements: Yann Richet, Institut de Radioprotection et de Sûreté Nucleaire.



Discretization on grid 50×50.

Conservative estimate at 95%;

Candidate sets dimension between 1659 and 2084;

Volume of conservative estimate: 17.36 (true volume 22.0). Sequential

Conclusion

- GP can be used for uncertainty quantification on sets;
- different types of estimates, depending on the final objective;
- Optimal quasi-realizations for excursion sets lower the computational cost of quantities based on set realizations;
- Conservative estimates:
  - sequential strategies to reduce uncertainty;
  - GanMC: benchmark study with other algorithms;
  - Currently developing R package ConservativeEstimates.

Thanks for your attention!

Conclusion

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Thanks for your attention!

References

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A D. and Bect, J. and Chevalier, C. and Ginsbourger, D. (2015). *Quantifying uncertainties on excursion sets under a Gaussian random field prior*. Accepted, JUQ hal-01103644v2

Bolin, D. and Lindgren, F. (2015). *Excursion and contour uncertainty regions for latent Gaussian models.* JRSS: B, 77(1):85-106.

Chevalier, C., Ginsbourger, D., Bect, J., and Molchanov, I. (2013). *Estimating and quantifying uncertainties on level sets using the Vorobev expectation and deviation with Gaussian process models.* mODa 10.

How to reduce the uncertainty on the estimate?

Stepwise uncertainty reduction: find a sequence of evaluation points $X_1, X_2, \dots$ that optimally reduces the expected uncertainty on the future estimate, i.e. given an initial design $\mathbf{X}_n$, select

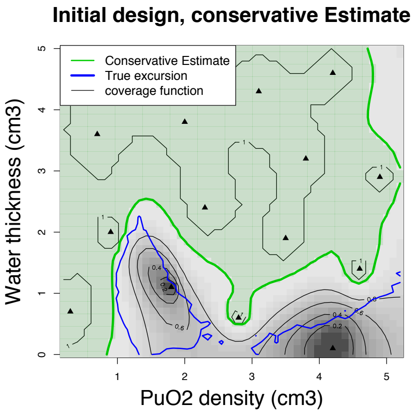
$$X_{n+1} \in \arg\min_{x_{n+1} \in D} \mathbb{E}_n[H_{n+1} \mid X_{n+1} = x_{n+1}]$$

Uncertainty function(s): many possible definitions, here

$$H_{n+1}^{\text{symm}} = \mathbb{E}_{n+1}[\mu(\Gamma \Delta Q_{\rho_{n+1}})], \quad \Gamma \Delta Q_{\rho_{n+1}} = \Gamma \setminus Q_{\rho_{n+1}} \cup Q_{\rho_{n+1}} \setminus \Gamma$$

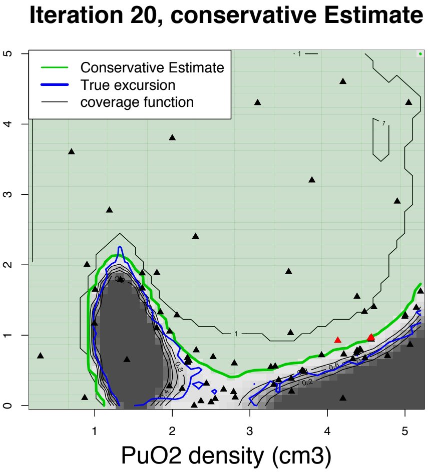
see [Bect et al. (2012), Chevalier et al. (2014)] and references therein for different definitions of H_{n+1}.

How to reduce the uncertainty on the estimate?



Criterion: $J_{n+q}^{\text{symm}}(\mathbf{x}_q) = \mathbb{E}_n[\mathbb{E}_{n+q}[\mu(\Gamma \setminus C_{\Gamma,n})] \mid \mathbf{X}_{n+q} = \mathbf{x}_q],$

Sequential strategies



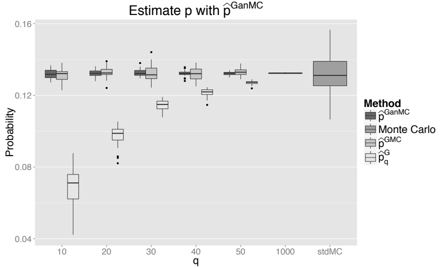
$n = 75$ new evaluations;

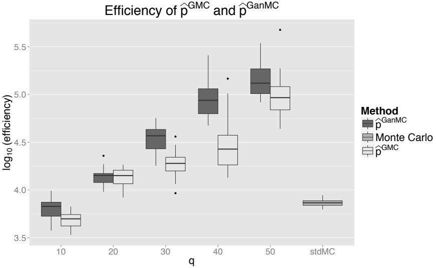
next evaluation chosen in order to minimize the future expected volume of the set difference $\Gamma \setminus Q_{\rho^*}$;

Volume of updated CE: 20.72 (true excursion: 22.0, old estimate: 17.36) Back

Joint work with: David Ginsbourger, Clément Chevalier, Julien Bect, Yann Richet.

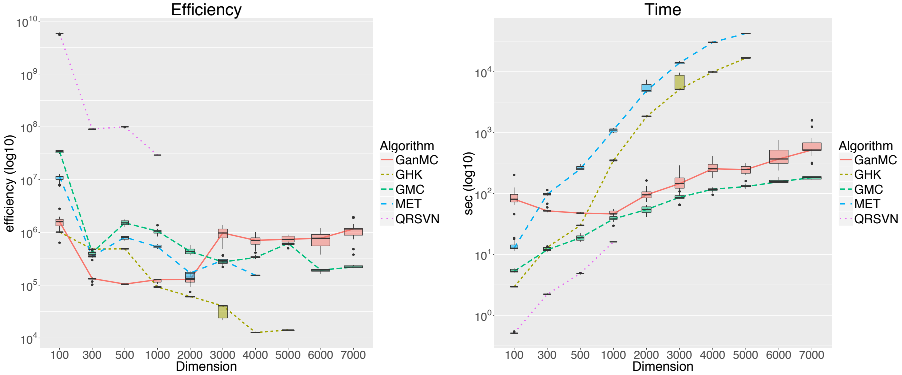
More comparisons anMC/MC





Benchmark: 6d GRF, discretization: 1000 Sobol' points, k Matérn ($\nu = 5/2$) with $\theta = [0.5, 0.5, 1, 1, 0.5, 0.5]^T$ and $\sigma^2 = 8$, m constant.

More comparisons anMC/MC



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