Identifying Dynamic Systems for Digital Twins of Engineering Assets

T. J. Rogers

September, 2023

PROBLEMS I CARE ABOUT

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A BRIEF ASIDE ON DIGITAL TWINS

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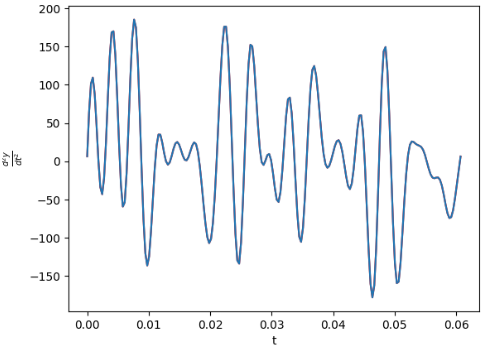
FITTING GPS TO DYNAMIC SYSTEMS



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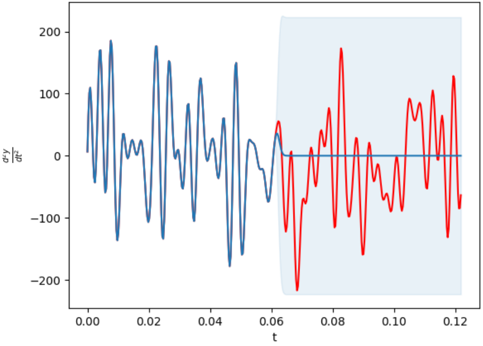

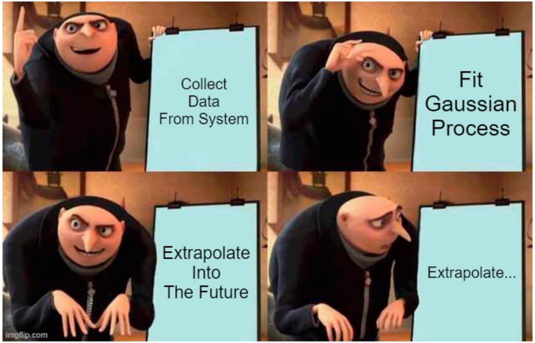

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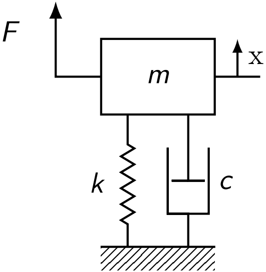

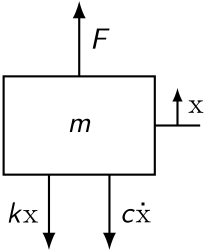

$$m\ddot{x} + c\dot{x} + kx = F$$

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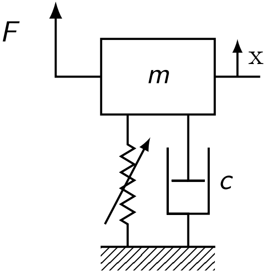
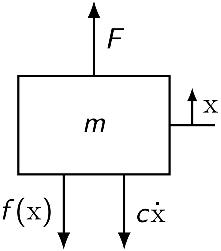
A QUICK REVIEW OF YEAR 1 MECHANICS AND...





$$m\ddot{x} + f(x, \dot{x}) = F$$

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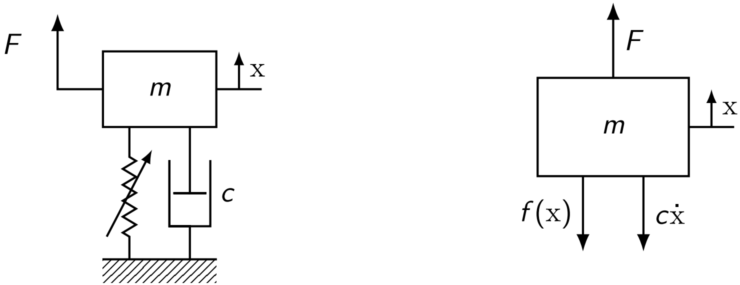


$$m\ddot{x} + f(x, \dot{x}) = F$$

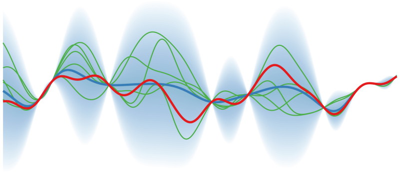
Which all works great if we can know the physics first. . .

Gaussian Processes:

Flexible nonlinear Bayesian regression.

$$y = f(x) + \varepsilon \quad \varepsilon \sim \mathcal{N}(0, \sigma_n^2)$$

$$f(x) \sim \mathcal{GP}(m(x), k(x, x'))$$



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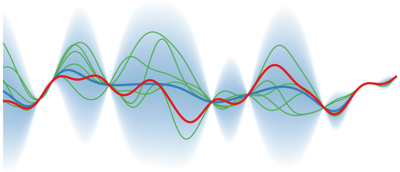

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The GP can be rewritten as an LGSSM (Hartikainen 2010) and similarly for the GP-LFM (Hartikainen 2012).

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

Temporal
Covariance
Function

$k\left(t, t'\right)$

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

Temporal Covariance Function		Covariance Spectral Density
$k(t, t')$	→	$S_k(\omega)$

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

Temporal Covariance Function		Covariance Spectral Density		Rational Transfer Function
$k(\mathrm{t}, \mathrm{t}')$	→	$S_k(\omega)$	→	$H(i\omega)$

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THE FUNDAMENTAL TRICK

Turn a Gaussian process ($\mathcal{O}(N^3)$) into a linear SSM ($\mathcal{O}(N)$)

Temporal Covariance Function	Covariance Spectral Density	Rational Transfer Function	Continuous Time SSM
$k\,(\mathrm{t},\mathrm{t}')$ →	$S_k\,(\omega)$ →	$H(i\omega)$ →	$F\ \mathrm{f}(t) + L\ w(t)$

Hartikainen, Jouni, and Simo Särkkä. "Kalman filtering and smoothing solutions to temporal Gaussian process regression models." 2010 *IEEE International Workshop on Machine Learning for Signal Processing.* IEEE.

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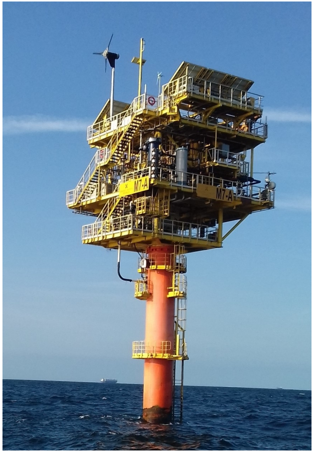
THREE USES AND EXTENSIONS

Three applications:

1. Recovering Unknown Forces
2. Extracting nonlinear components of SDOF ODEs
3. Some promising extensions for more interesting scenarios



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Problem 1: Unknown Loads

LATENT FORCES ARE NATURAL SOLUTIONS

$$M\ddot{x} + C\dot{x} + Kx = U$$

LATENT FORCES ARE NATURAL SOLUTIONS

$$M\ddot{\mathrm{x}} + C\dot{\mathrm{x}} + K\mathrm{x} = U$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix} U$$

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$$U \sim \mathcal{GP}\left(0, k(\mathrm{t}, \mathrm{t}')\right)$$

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$$U \sim \mathcal{GP}\left(0, k(\mathrm{t}, \mathrm{t}')\right)$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{f} \\ \ddot{f} \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -M^{-1}K & -M^{-1}C & M^{-1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda^2 & -2\lambda \end{pmatrix} \begin{bmatrix} x \\ \dot{x} \\ u \\ \dot{u} \end{bmatrix} + Lw(t)$$

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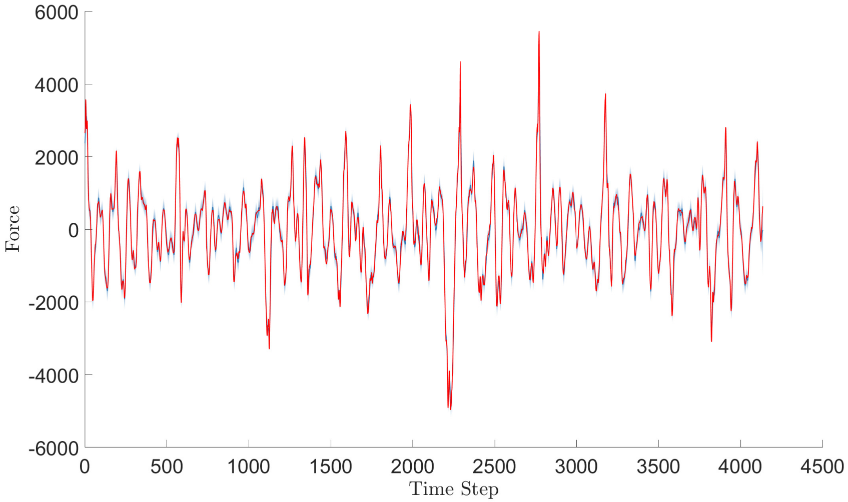
MDOF EXAMPLE

NMSE=1.05%



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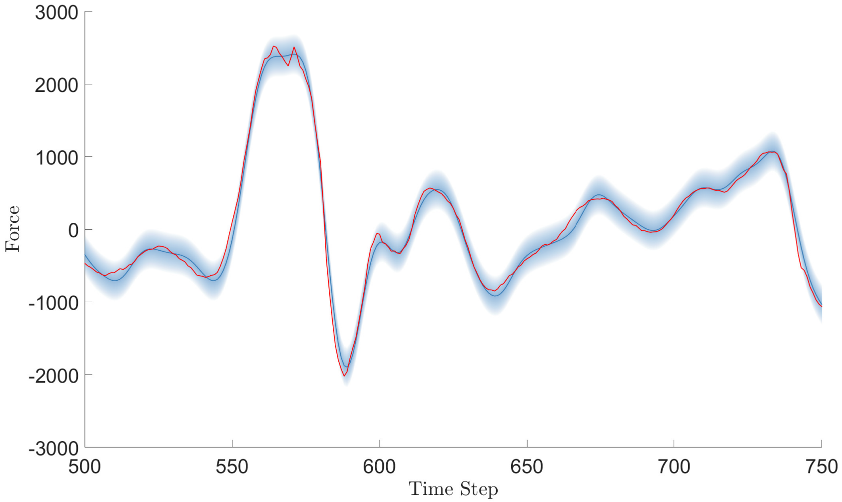
MDOF EXAMPLE



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Problem 2: Learning Nonlinear SDOF ODEs

Let's consider a nonlinear system:

$$m\ddot{x} + c\dot{x} + kx + f(x, \dot{x}) = U$$

Masri and Caughey introduced the Restoring Force Surface method (1979),

$$m\ddot{x} + c\dot{x} + kx = U - f(x, \dot{x})$$

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California Institute of Technology,
Pasadena, Calif. 91106. Mem. ASME

A Nonparametric Identification Technique for Nonlinear Dynamic Problems

A nonparametric identification technique is presented that uses information about the state variables of nonlinear systems to express the system characteristics in terms of orthogonal functions. The method can be used with deterministic or random excitation (stationary or otherwise) to identify dynamic systems with arbitrary nonlinearities, including those with hysteretic characteristics. The method is shown to be more efficient than the Weiner-kernel approach in identifying nonlinear dynamic systems of the type considered.

Introduction

The identification of dynamic system models through the use of experimental data is a problem of considerable importance in the area of applied mechanics. The development of procedures to handle the problem has received wide attention in recent years because of the development of efficient computer-oriented systems identification techniques and the availability of sophisticated experimental apparatus for accurate, convenient gathering and analysis of test data.

The approaches used to handle different identification problems and the degree of difficulty in identification depend on the classification of the case:

1 Linear/nonlinear.
2 Stationary/nonstationary.
3 Discrete/continuous.
4 Single-input/multi-input.
5 Deterministic/stochastic.
6 The degree of *a priori* knowledge about the system [1–22].

System identification methods can also be classified on the basis of their search space: (*a*) parametric methods that search in parameter space and (*b*) nonparametric methods that search in function space.

Presented at the Eighth U.S. National Congress of Applied Mechanics, University of California at Los Angeles, Los Angeles, Calif., June 26–30, 1978.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until September 1, 1979. Readers who need more time to prepare a Discussion should request an extension of the deadline from the Editorial Department. Manuscript received by ASME Applied Mechanics Division, July, 1978.

Basically, parametric methods seek to determine the value of parameters in an assumed model of the system to be identified, while nonparametric methods produce the best functional representation of the system without *a priori* assumptions about the system model.

So far, most of the identification work in applied mechanics has been parametric. A considerable amount of effort has been devoted to determining efficient algorithms and techniques for estimating the magnitude of various parameters in an assumed mathematical model of the dynamic system of interest. One of the limitations of this class of methods is that the type of model, once assumed at the onset of the investigation, cannot be changed. Thus, if the type of model does not closely resemble the characteristics of the physical system (which, in many practical dynamic problems, are not fully understood), the prediction of the future behavior of the identified system may be in substantial error.

The restriction of forcing the system characteristics to fit an assumed form can be eliminated by using nonparametric identification techniques such as the ones that use the Volterra-series, or Weiner-kernel, approach [23–25]. However, this approach has its own problems of

1 Greater mathematical complexity.
2 Serious difficulties with convergence rate.
3 Excessive computation time.
4 Very demanding (and usually unrealistic) storage requirements.
5 Restrictions on the nature of dynamic systems to be identified (nonhysteretic, stationary), and on the input signals that can be used (white noise).

In an effort to alleviate some of these problems, this paper presents a relatively simple and straightforward approach to identify a broad

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Unfortunately, if the displacement and velocity isn't known this can be hard.

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Journal of Applied Mechanics JUNE 1979, VOL. 46 / **433**

POSING RESTORING FORCES AS A LATENT FORCE MODEL

GP Latent Force Problem:

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$$m\ddot{x} + c\dot{x} + kx = U \quad U \sim \mathcal{GP}\left(0, k(t, t')\right)$$

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Latent Restoring Forces:

Model the restoring forcing of the nonlinear system:

$$m\ddot{x} + c\dot{x} + kx + f(x, \dot{x}) = U$$

As a GP in time,

$$m\ddot{x}+c\dot{x}+kx+R = U \quad R \sim \mathcal{GP}\left(0, k(t, t')\right)$$

Solve as a **linear** state-space model.

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Solve as a **linear** state-space model.

We will use the Duffing oscillator as a test case:

$$m\ddot{x} + c\dot{x} + kx + k_3 x^3 = U$$

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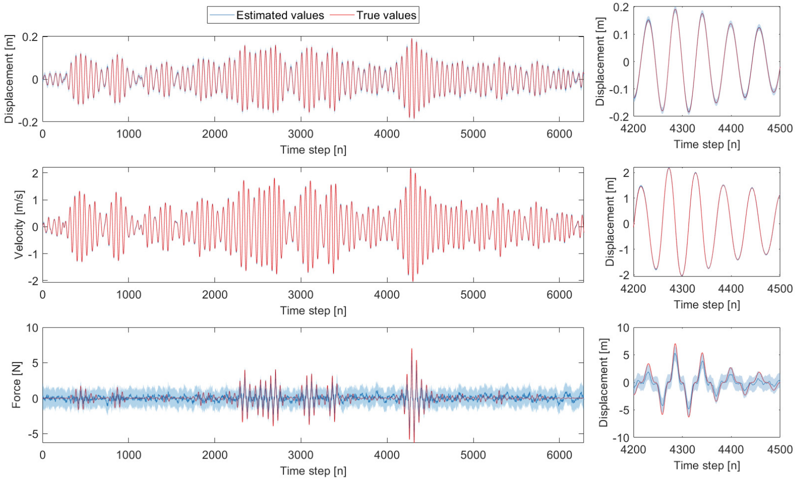
DOES IT WORK?



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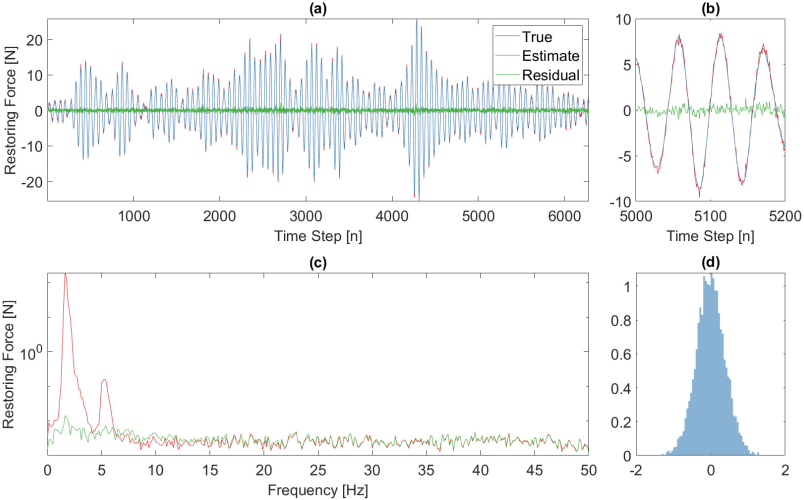

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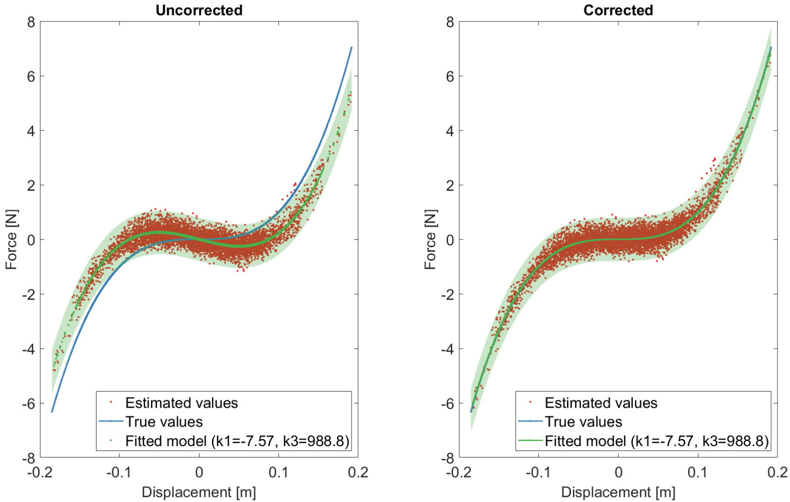

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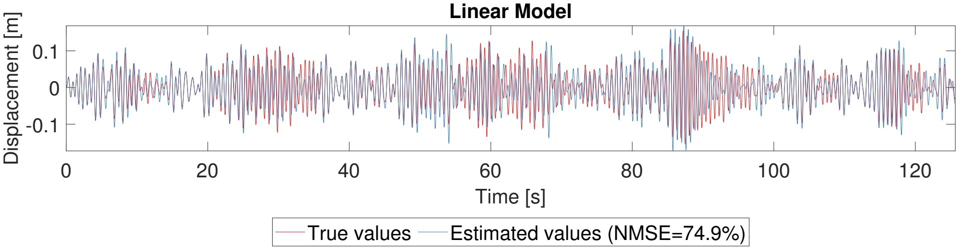

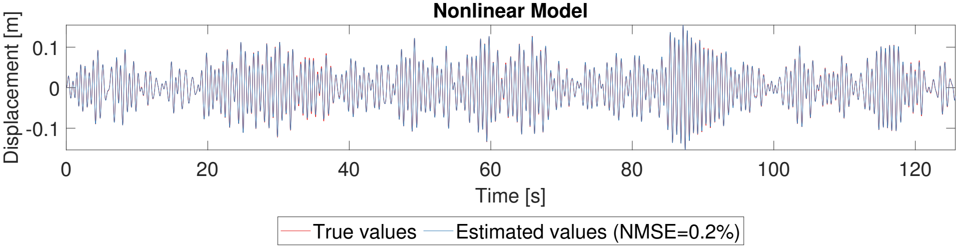

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Problem 3: Promising Extensions

UNABLE TO ACCESS INPUTS

$$m\ddot{x} + c\dot{x} + kx + f(x, \dot{x}) = U$$

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UNABLE TO ACCESS INPUTS

$$m\ddot{x} + c\dot{x} + kx + f(x, \dot{x}) = U$$

Assume,

$$f(x, \dot{x}) \approx R(t) \sim \mathcal{GP}\left(0, k_f(t, t')\right), \quad U \sim \mathcal{GP}\left(0, k_u(t, t')\right)$$

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Then,

$$\hat{R}(t) = U - R(t) = \mathcal{GP}\left(0, \hat{k}(t, t')\right)$$

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Then,

$$\hat{R}(t) = U - R(t) = \mathcal{GP}\left(0, \hat{k}(t, t')\right)$$

Take expectations to separate $R(t)$ and U.

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\hat{r}} \\ \ddot{\hat{r}} \end{bmatrix} = \begin{bmatrix} F_{sys} & B_{sys} \\ 0 & F_{\hat{R}} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \hat{r} \\ \dot{\hat{r}} \end{bmatrix} + L\nu(t)$$

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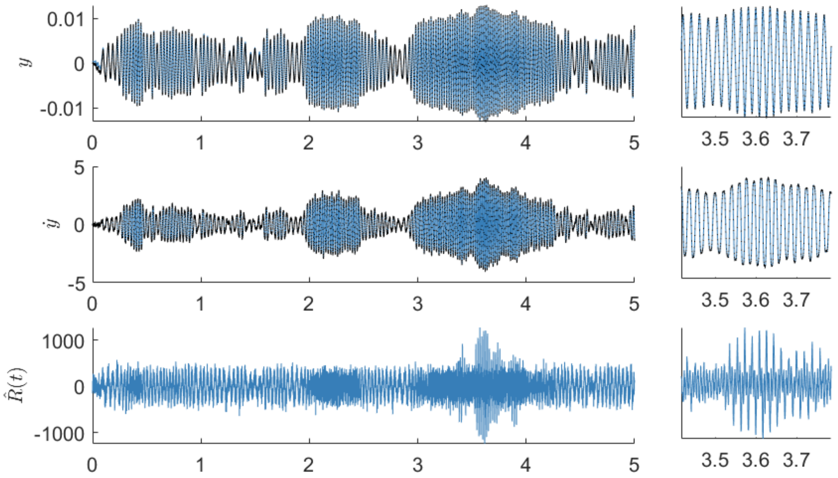
STATE ESIMTATION FOR OUTPUT-ONLY NONLINEAR SYSTEM



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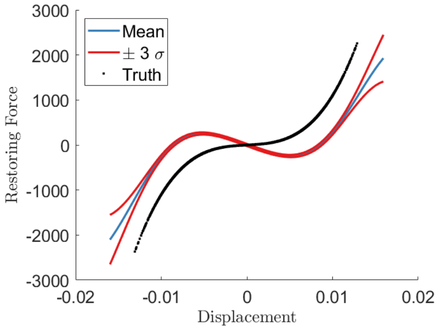
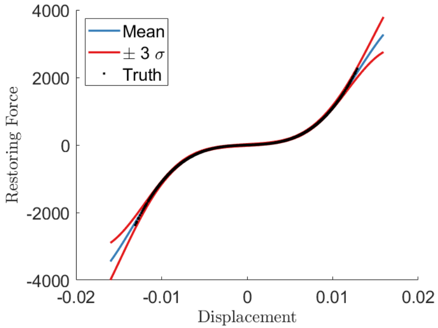
ACCESSING THE NONLINEARITY





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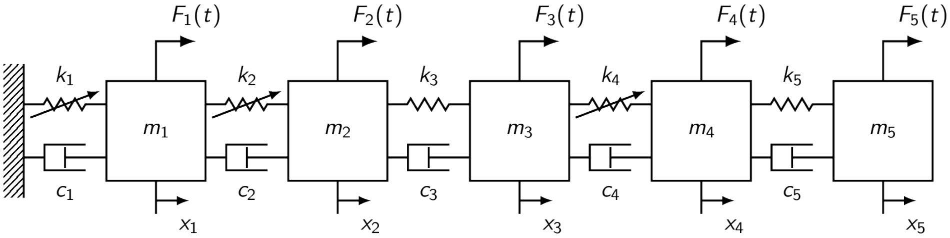
EXTENSIONS TO MORE DEGREES OF FREEDOM



IMPOSE PHYSICALLY MEANINGFUL STRUCTURE

We have the potential for model discrepancy between each degree of freedom.

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{r}} \\ \ddot{\boldsymbol{r}} \end{bmatrix} = \begin{bmatrix} F_{sys} & B_{sys} \\ 0 & F_R \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{r}} \\ \ddot{\boldsymbol{r}} \end{bmatrix} + L\nu(t)$$

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Use Newton's third law to impose structure in B_{sys}.

$$B_{sys,xr} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

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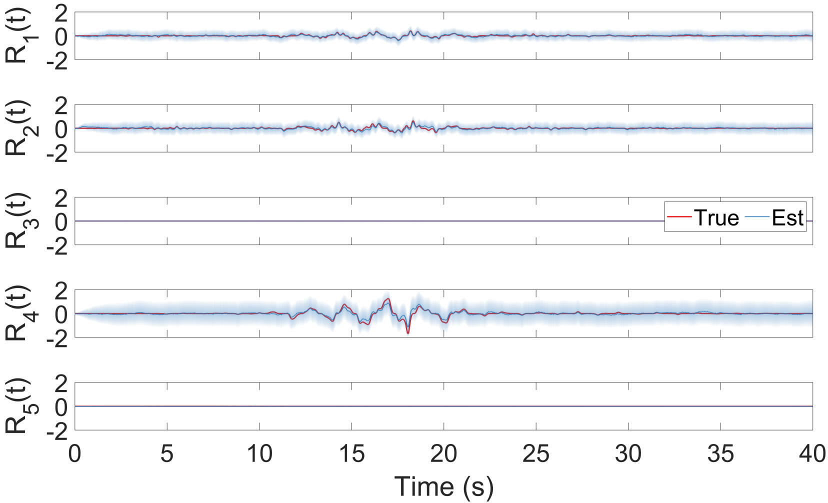
DETECTION AND LOCALISATION



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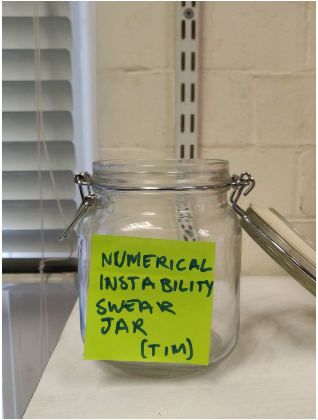
WHAT SHOULD BE A CONCLUSION

A word of warning. . .



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SOME REFERENCES

1. Rogers, T. J., Worden, K., & Cross, E. (2020). *On the application of Gaussian process latent force models for joint input-state-parameter estimation: With a view to Bayesian operational identification.* Mechanical Systems and Signal Processing, 140. doi:10.1016/j.ymssp.2019.106580
2. Rogers, T. J., & Friis, T. (2022). *A latent restoring force approach to nonlinear system identification.* Mechanical Systems and Signal Processing, 180. doi: 10.1016/j.ymssp.2022.109426
3. Longbottom, J.D., Cross, E.J. & Rogers, T.J. (2022) *Output-only Bayesian semi-parametric identification of a nonlinear dynamic system.* In Proceedings 9th International Operational Modal Analysis Conference (pp. 186-194).

The University Of Sheffield.

Identifying Dynamic Systems for Digital Twins of Engineering Assets

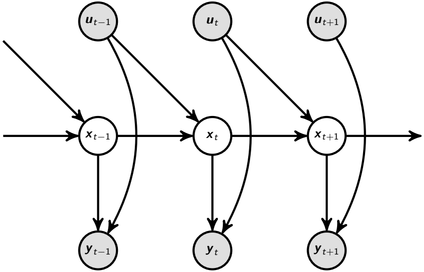
T. J. Rogers

September, 2023

NONLINEAR SSMS

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 = u$$

$$x_t \sim f_\theta\left(x_t \,\middle|\, x_{t-1}, u_{t-1}\right)$$
$$y_t \sim g_\theta\left(y_t \,\middle|\, x_t, u_t\right)$$



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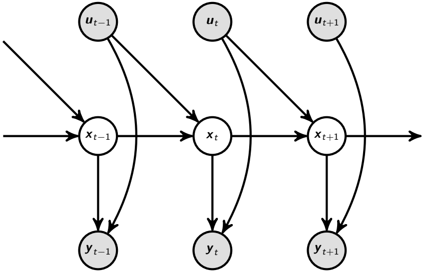
NONLINEAR SSMS

$$m\ddot{x} + c\dot{x} + kx + k_3 x^3 = u \qquad u \sim \mathcal{GP}\left(0, k(t, t')\right)$$

$$x_t \sim f_\theta\left(x_t \,\middle|\, x_{t-1}, u_{t-1}\right)$$
$$y_t \sim g_\theta\left(y_t \,\middle|\, x_t, u_t\right)$$



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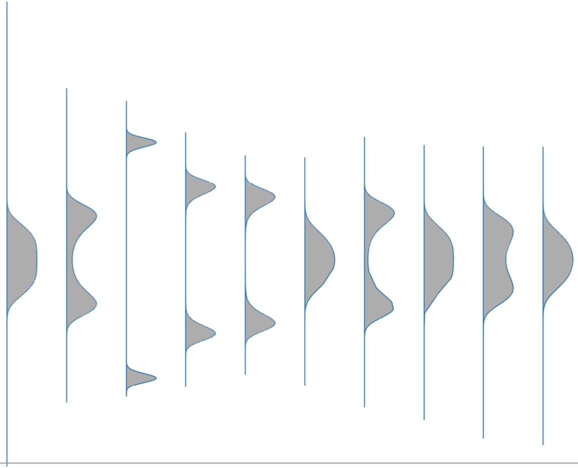
PARTICLE FILTERING

Estimating sequences of probability distributions:



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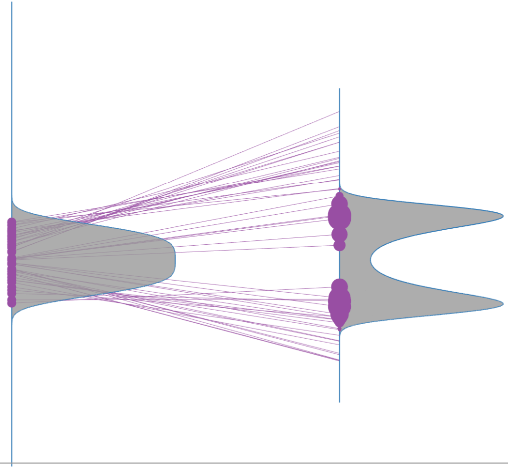
PARTICLE FILTERING

Particle Propagation and Weighting:



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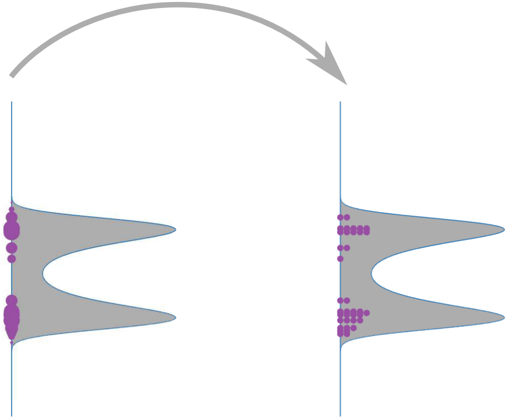
PARTICLE FILTERING

Resampling:



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PARTICLE SMOOTHING

A slight complication...this gives the filtering distribution but really want the smoothing distribution $p(x_{1:T} \mid y_{1:T})$.

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PARTICLE SMOOTHING

A slight complication. . . this gives the filtering distribution but really want the smoothing distribution $p\left(x_{1:T} \mid y_{1:T}\right)$.

We will use an MCMC scheme to sample from this using the particle filter.

In particular we use a Particle Gibbs with Ancestor Sampling approach (Lindsten 2014) to sample from $p\left(x_{1:T} \mid y_{1:T}\right)$.

Details:

Rogers, Timothy J., Worden, Keith and Cross Elizabeth J.. "Bayesian Joint Input-State Estimation for Nonlinear Systems." *Vibration 3.3 (2020): 281-303.*

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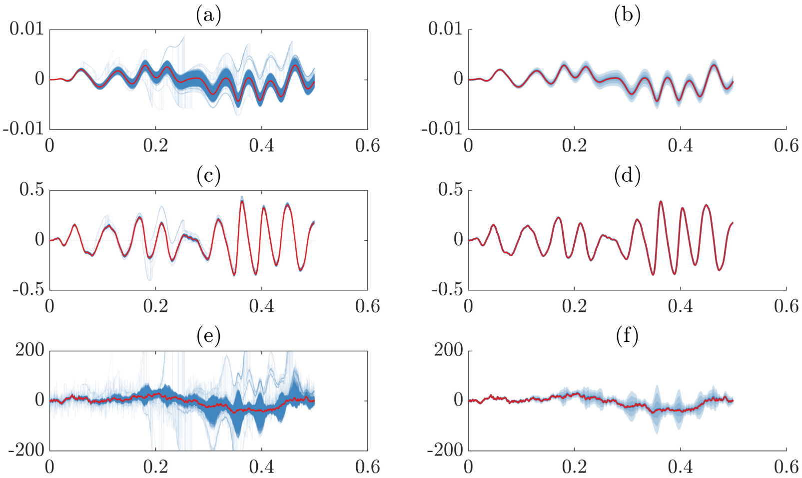
A DUFFING EXAMPLE



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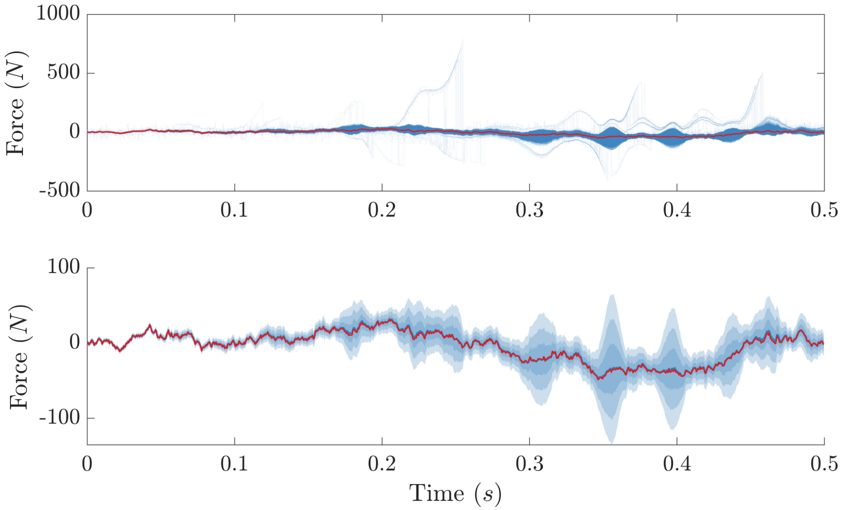

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